Method of Variation of parameters.

$$(AD^2+BD+C)y = X \rightarrow (1)$$

$$C.F = c_1y_1 + c_2y_2.$$

To find P.I

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

$$P.I. = Py_1 + Qy_2.$$

where $P = -\int \frac{y_2 X}{W} dx$

& $Q = \int \frac{y_1}{W} X \, dx.$

Problems

(6) Using the method of variation of parameters, solve

(a) $(D^2-6D+9)y = e^{2x}$.

Solution: Given equation is

$$(D^2-6D+9)y = e^{2x} \rightarrow (1)$$

To find C.F.

Consider $(D^2-6D+9)y=0$.

A.E. is $m^2-6m+9=0$

$\Rightarrow m = 3, 3.$

$\Rightarrow (Ax+B)e^{mx}$

$\therefore C.F = y = (Ax+B)e^{3x} \rightarrow (2)$

$\Rightarrow \boxed{y = (Ax+B)e^{3x}}$

To find P.I.

Now $C.F = Axe^{3x} + Be^{3x}$.

$\Rightarrow y_1 = xe^{3x}$; $y_2 = e^{3x}$.

$y_1 = xe^{3x}$; $y_2 = e^{3x}$

$\Rightarrow y_1' = 3xe^{3x}+e^{3x}$; $y_2' = 3e^{3x}$.

$$\therefore W = \begin{vmatrix} xe^{3x} & e^{3x} \\ 3xe^{3x}+e^{3x} & 3e^{3x} \end{vmatrix}.$$

$$= \cancel{3x(e^{3x})^2} - \cancel{3x(e^{3x})^2} - (e^{3x})^2.$$

$$= -e^{6x}$$

$\Rightarrow \boxed{W = -e^{6x}}$ $\boxed{X = e^{2x}}$

$$P.I. = Py_1 + Qy_2 \rightarrow (3)$$

Now $P = -\int \frac{y_2 X}{W} dx.$

$$= -\int \frac{e^{3x} \cdot e^{2x}}{-e^{6x}} dx.$$

$$= \int e^{-x} dx = -e^{-x}.$$

$$Q = \int \frac{y_1 X}{W} dx.$$

$$= \int \frac{xe^{3x} \cdot e^{2x}}{-e^{6x}} dx = -\int xe^{-x} dx.$$

$$= -\left[x\left(\frac{e^{-x}}{-1}\right) - 1 \cdot (e^{-x})\right] = xe^{-x} + e^{-x}.$$

Now (3) $\Rightarrow$

$$P.I. = -e^{-x} \cdot xe^{3x} + (xe^{-x}+e^{-x})e^{3x}.$$

$$= -xe^{2x} + xe^{2x} + e^{2x}.$$

$$= e^{2x}.$$

$$\therefore \boxed{y = Axe^{3x} + Be^{3x} + e^{2x}}$$

is the solution.

6(b) $\frac{d^2y}{dx^2} + 4y = \tan 2x.$

Solution: $X = \tan 2x.$

Given: $(D^2+4)y = \tan 2x. \rightarrow (1)$

To find C.F.

$(D^2+4)y = 0$

A.E is $m^2+4=0.$

$m = \pm 2i = \alpha \pm i\beta.$

$\alpha = 0,\ \beta = 2.$

$C.F. = C_1 \cos 2x + C_2 \sin 2x.$

$= c_1 y_1 + c_2 y_2.$

$y_1 = \cos 2x \ ; \ y_2 = \sin 2x$

$\Rightarrow y_1' = -2\sin 2x \ ; \ y_2' = 2\cos 2x.$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}.$

$= 2\cos^2 2x + 2\sin^2 2x = 2.$

$\Rightarrow \boxed{W = 2} \quad X = \tan 2x.$

$P.I. = Py_1 + Qy_2 \rightarrow (2)$

$P = -\int \frac{y_2 X}{W} dx.$

$= -\int \frac{\sin 2x \cdot \tan 2x}{2} dx.$

$= -\frac{1}{2} \int \frac{\sin^2 2x}{\cos 2x} dx.$

$= -\frac{1}{2} \int \frac{1-\cos^2 2x}{\cos 2x} dx.$

$= -\frac{1}{2} \int (\sec 2x - \cos 2x) dx.$

$= -\frac{1}{2} \frac{\log(\sec 2x + \tan 2x)}{2} + \frac{\sin 2x}{4} + A.$

$Q = \int \frac{y_1 X}{W} dx.$

$= \int \frac{\cos 2x \tan 2x}{2} dx$

$= \frac{1}{2} \int \sin 2x \, dx.$

$= -\frac{\cos 2x}{4} + B.$

$\therefore (2) \Rightarrow$

$y = \left[-\frac{1}{4}\log(\sec 2x + \tan 2x) + \frac{\sin 2x}{4} + A\right] \cos 2x +$

$\left[-\frac{\cos 2x}{4} + B\right] \sin 2x.$

$= -\frac{\cos 2x \log(\sec 2x + \tan 2x)}{4} + \cancel{\frac{\sin 2x \cos 2x}{4}} + A\cos 2x - \cancel{\frac{\sin 2x \cos 2x}{4}} + B\sin 2x.$

$= A\cos 2x + B\sin 2x - \frac{\cos 2x \log(\sec 2x + \tan 2x)}{4}.$

6 (h) $y'' - 2y' + 2y = e^x \tan x$

Solution

$(D^2 - 2D + 2)y = 0$.

$m = 1 \pm i \quad X = e^x \tan x$.

C.F. $y = c_1 e^x \cos x + c_2 e^x \sin x$.
$= c_1 y_1 + c_2 y_2$

$y_1 = e^x \cos x \quad ; \quad y_2 = e^x \sin x$.

$\Rightarrow y_1' = +e^x \cos x - e^x \sin x; \quad y_2' = e^x \cos x + e^x \sin x$.

Now $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= \begin{vmatrix} e^x \cos x & e^x \sin x \\ -e^x \sin x + e^x \cos x & e^x \cos x + e^x \sin x \end{vmatrix}$

$= e^x \cos x \cdot e^x \cos x + e^x \cos x e^x \sin x - e^x \cos x e^x \sin x$

$+ e^x \sin x e^x \sin x - e^x \cos x e^x \sin x$

$= e^{2x} \cos^2 x + e^{2x} \sin^2 x = e^{2x}$.

$P = -\int \frac{y_2 X}{W} dx$.

$= -\int \frac{e^x \sin x \cdot e^x \tan x}{e^{2x}} dx$.

$= -\int \frac{\sin^2 x}{\cos x} dx = -\int \frac{1 - \cos^2 x}{\cos x} dx$

$= -\log(\sec x + \tan x) + \sin x + A$.

$Q = \int \frac{e^x \cos x \, e^x \tan x}{e^{2x}} dx$.

$= \int \sin x \, dx = -\cos x + B$.

$y = Ae^x \cos x + Be^x \sin x -$
$e^x \cos x \log(\sec x + \tan x)$.

---

6 (i) $y'' - 2y' + y = e^x \log x$

Soln:

C.F. $= c_1 x e^x + c_2 e^x$, $W = -e^{2x}$.

P.I. $= P y_1 + Q y_2$.

$= -xe^x \int \frac{e^x e^x \log x}{-e^{2x}} dx +$

$e^x \int \frac{x e^x e^x \log x}{-e^{2x}} dx$.

$= xe^x \int \log x \, dx - e^x \int x \log x \, dx$.

$= xe^x \left[x \cdot \log x - \int x \cdot \frac{1}{x} dx \right]$

$- e^x \left[\log x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx \right]$.

$= xe^x (x \log x - x) - e^x \left(\frac{x^2}{2} \log x - \frac{x^2}{4} \right)$

$= x^2 e^x \log x - x^2 e^x - \frac{x^2}{2} e^x \log x - \frac{x^2 e^x}{4}$.

$= \frac{x^2 e^x \log x}{2} - \frac{3}{4} x^2 e^x$.

---

6 (j) $(D^2 + a^2)y = \tan ax$

$m = \pm ia$, C.F. $= A \cos ax + B \sin ax$

$W = a$

$P = \frac{1}{a^2} \log(\sec ax + \tan ax) + \frac{\sin ax}{a^2}$.

$Q = -\frac{\sin ax \cos ax}{a^2}$

P.I. $= -\frac{\cos ax}{a^2} \log(\sec ax + \tan ax)$.

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(6) (c) $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$.

Solution:

C.F. $(D^2 - 6D + 9)y = 0$

$m = 3, 3$.

C.F. $y = c_1 x e^{3x} + c_2 e^{3x}$

$= c_1 y_1 + c_2 y_2$

$W = -e^{6x}$, $X = e^{3x} \cdot x^{-2}$

P.I. $= Py_1 + Qy_2$.

$= \left[-\int \frac{e^{3x} \cdot e^{3x} x^{-2}}{-e^{6x}} dx\right] xe^{3x} +$

$\left[\int \frac{xe^{3x} \cdot e^{3x} \cdot x^{-2}}{-e^{6x}} dx\right] e^{3x}$.

$= xe^{3x}\int x^{-2} dx - e^{3x}\int x^{-1} dx$.

$= xe^{3x}\left[\left(\frac{x^{-1}}{-1}\right) + A\right] - e^{3x}[\log x + B]$.

$= \left(-\frac{1}{x} + A\right) xe^{3x} + (-\log x + B)e^{3x}$.

$\Rightarrow y = Axe^{3x} + Be^{3x} - e^{3x} - e^{3x}\log x$.

(d) $\frac{d^2y}{dx^2} + y = \tan x$.

Solution: $y = c_1 \cos x + c_2 \sin x$

$= c_1 y_1 + c_2 y_2$

$W = 1$, P.I. $= Py_1 + Qy_2$

$X = \tan x$.

$P = -\int \frac{\sin x \tan x}{1} dx$

$= -\log(\sec x + \tan x) + \sin x + A$

$Q = \int \frac{\cos x \tan x}{1} dx = -\cos x + B$.

$\therefore y = \cos x[P] + \sin x[Q]$

$\Rightarrow y = A\cos x + B\sin x - \cos x \log(\sec x + \tan x)$

(6)(e) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x}$.

Solution: $X = x^{-1}e^x$,

C.F. $= c_1 x e^x + c_2 e^x = c_1 y_1 + c_2 y_2$

$W = -e^{2x}$.

$P = \log x + A$; $Q = -x + B$.

$y = (\log x + A)xe^x + (-x + B)e^x$.

$\Rightarrow y = Axe^x + Be^x + xe^x \log x - xe^x$.

6(f) $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$.

Solution: $X = \operatorname{cosec} x$, $m = \pm i$.

C.F. $= y = c_1 \cos x + c_2 \sin x = c_1 y_1 + c_2 y_2$

$W = 1$, P.I. $= Py_1 + Qy_2$

$P = -x + A$, $Q = \log \sin x + B$.

$\therefore y = \cos x(-x + A) + \sin x(\log \sin x + B)$

$\Rightarrow y = A\cos x + B\sin x - x\cos x + \sin x \log \sin x$

(g) $\frac{d^2y}{dx^2} + a^2 y = \sec ax$.

Solution: $X = \sec ax$, $y = c_1 \cos ax + c_2 \sin ax$

$W = a$

$P = -\int \frac{\sin ax \sec ax}{a} dx = -\frac{1}{a}\int \tan ax \, dx$.

$P = \frac{1}{a^2} \log \sec ax + A$.

$Q = \int \frac{\cos ax \sec ax}{a} dx = \frac{x}{a} + B$.

$y = A\cos ax + B\sin ax - \frac{1}{a^2}\log \sec ax \cos ax + \frac{x}{a}\sin ax$.

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6 (k) $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$.

Solution: $X = \frac{2}{1+e^x}$, $m = \pm 1$

C.F. $y = c_1 e^x + c_2 e^{-x}$

$y_1 = e^x$ $\quad y_2 = e^{-x}$, $W = -2$

$P = -\int \frac{e^{-x} \cdot \frac{2}{1+e^x}}{-2} dx$.

$= \int \frac{e^{-x}}{1+e^x} dx = \int \frac{1}{e^x(1+e^x)} dx \longrightarrow (1)$

Now

$\frac{1}{e^x(1+e^x)} = \frac{A}{e^x} + \frac{B}{1+e^x}$

$1 = A(1+e^x) + Be^x$.

Coeff of $e^x \Rightarrow A + B = 0$.

Constant $\Rightarrow \boxed{A=1} \Rightarrow \boxed{B=-1}$

$\therefore (1) \Rightarrow$

$P = \int \frac{1}{e^x} dx - \int \frac{1}{1+e^x} dx$.

$= \int e^{-x} dx - \int \frac{dx}{1+e^x}$,

$= -e^{-x} - \int \frac{1+e^x - e^x}{1+e^x} dx$.

$= -e^{-x} - \int 1 dx + \int \frac{e^x}{1+e^x} dx$.

$= -e^{-x} - x + \log(1+e^x)$.

$Py_1 = e^x\left(-e^{-x} - x + \log(1+e^x)\right)$

$= -1 - xe^x + e^x \log(1+e^x)$.

$Q = \int \frac{e^x \cdot \frac{2}{1+e^x}}{2} dx$.

$= \int \frac{e^x}{1+e^x} dx = \log(1+e^x)$

$Q y_2 = \left[\log(1+e^x)\right] e^{-x}$

$= e^{-x} \log(1+e^x)$

$\therefore y = c_1 e^x + c_2 e^{-x} - 1 - xe^x$
$+ (e^x - e^{-x}) \log(1+e^x)$

6 (l) $\frac{d^2y}{dx^2} + y = \sec x$.

Soln: $y'' + y = \sec x$

$(D^2+1)y = \sec x$.

$m = \pm i$, $X = \sec x$.

CF $= A\cos x + B\sin x = c_1y_1 + c_2y_2$

$W = 1$

$P = \log \sec x$, $Q = x$

P.I. $= \cos x \log \sec x + x \sin x$.

6 (m) $y'' + y = \sec^2 x$.

Solution: $P = -\int \frac{\sin x \sec^2 x}{1} dx$

$= -\sec x$.

$Q = \int \cos x \sec^2 x \, dx = \int \sec x \, dx$

$= \log(\sec x + \tan x)$

P.I. $= -\cos x \sec x + \sin x \log(\sec x + \tan x)$.

6 (n) $(D^2+4)y = 4\tan 2x$

$W = 2$

P.I. $= -\cos 2x \log(\sec 2x + \tan 2x)$

6 o) $(D^2+4)y = \operatorname{cosec} 2x$.

Solution: $m = \pm 2i$

$CF = A\cos 2x + B\sin 2x, W = 2$.

$P = -\frac{1}{2}x, \quad Q = \frac{1}{2}\log\sin 2x$.

$P.I. = \frac{1}{2}\left[-x\cos 2x + \sin 2x + \log\sin 2x\right]$

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6) p) $(D^2+1)y = x\sin x$.

Solution: $m = \pm i$

$CF = A\cos x + B\sin x$.

$W = 1, \quad X = x\sin x$.

$P = -\int \frac{\sin x \, x\sin x}{1} dx$.

$= -\int x\sin^2 x dx$

$= -\int x\left(\frac{1-\cos 2x}{2}\right)dx$.

$= -\frac{1}{2}\int (x - x\cos 2x)dx$

$= -\frac{1}{2}\int x\cos 2x \, dx$.

$= -\frac{x^2}{4} + \frac{1}{2}\int x\cos 2x \left(\frac{\sin 2x}{2}\right) - (1)\left(-\frac{\cos 2x}{4}\right)\Big]$

$= -\frac{x^2}{4} + \frac{1}{2}\left[x\left(\frac{\sin 2x}{2}\right)\right.$

$+ \frac{x\sin 2x}{4} + \frac{\cos 2x}{8}$.

$= -\frac{x^2}{4}$

$Q = \int \frac{\cos x \cdot x\sin x}{1} dx$

$= \int \frac{x \cdot \sin 2x}{2} dx$.

$= \frac{1}{2}\left[x\left(-\frac{\cos 2x}{2}\right) - 1\left(-\frac{\sin 2x}{4}\right)\right]$

$= -\frac{x\cos 2x}{4} + \frac{\sin 2x}{8}$

$P.I. = \cos x \cdot P + \sin x \cdot Q$.

$=$

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6 q) $y'' + 4y = 4\sec^2 2x$.

~~Solution:~~

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6 r) $(D^2+9)y = \sec 3x$

6 s) $(D^2-4)y = e^{2x}$

6 t) $y'' + 2y' + y = e^{-x}\cos x$.

6 u) $(D^2+16)y = \sec 4x$.

6 v) $(D^2+16)y = 4\tan 4x$.

6 w) $D^2 - 2D = e^x\sin x$.

6(x) $(D^2-D)y = e^x\sin x$.

Solution: $m = 0, 1 \Rightarrow CF$ is $y = A + Be^x$.
$= c_1y_1 + c_2y_2$

$W = e^x, \quad y_1 = 1, y_2 = e^x$.

$X = e^x\sin x$.

$P = -\int \frac{e^x \cdot e^x\sin x}{e^x} dx$.

$= -\int e^x\sin x \, dx$,

$= -\frac{e^x}{2}(\sin x - \cos x)$

$Q = \int \frac{1 \cdot e^x\sin x}{e^x} dx = \int \sin x \, dx$

$= -\cos x$,

$P.I. = Py_1 + Qy_2$

$= \frac{e^x}{2}(\cos x - \sin x) \cdot 1 + (-\cos x)e^x$.

$= \frac{e^x\cos x}{2} - \frac{e^x\sin x}{2} - e^x\cos x$.

$= -\frac{e^x}{2}(\cos x + \sin x)$.

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07/09/2020

II B.SC MATHEMATICS

DIFFERENTIAL EQUATIONS

UNIT-II Contd.....

Equations reducible to linear equations with Constant Coefficients.

There are two methods of linear differential equations with variable Coefficients, which can be reduced to linear D.E. with Constant Coefficients by suitable substitutions.

1. Cauchy's homogeneous linear equation. [Cauchy - Euler equation]

An equation of the form

$$x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + k_{n-1} x \frac{dy}{dx} + k_n y = X \rightarrow (1)$$

where X is a function of x, is called cauchy's homogeneous linear equation.

When $n=2$, (1) becomes

$$a x^2 \frac{d^2 y}{dx^2} + b x \frac{dy}{dx} + cy = X \rightarrow (2)$$

Then the definition is

Definition: An equation of the form $ax^2 \frac{d^2 y}{dx^2} + bx \frac{dy}{dx} + cy = X$ where a, b, c are Constants and X is a function of x, is called Cauchy's homogeneous linear equation.

Such equations can be reduced to linear D.Es. with Constant Coefficients by putting

$x = e^z \Rightarrow z = \log x$.

If $D' = \frac{d}{dz}$, then

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x}$$

$$z = \log x \Rightarrow \frac{dz}{dx} = \frac{1}{x}$$

$$\Rightarrow x\frac{dy}{dx} = \frac{dy}{dz} = D'y$$

$$\Rightarrow x\frac{dy}{dx} = D'y \rightarrow (3)$$

Now $\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$.

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right).$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}.$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2 y}{dz^2}\left(\frac{1}{x}\right).$$

$$= \frac{1}{x^2}\left(\frac{d^2 y}{dz^2} - \frac{dy}{dz}\right).$$

$$\Rightarrow x^2\frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} - \frac{dy}{dz}.$$

$$= (D'^2 - D')y.$$

$$= D'(D'-1)y.$$

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Now $D = \frac{d}{dx}$, $D' = \frac{d}{dz}$

$\Rightarrow \boxed{xD = D'}$, $x^2D^2 = D'(D'-1)$,

$x^3D^3 = D'(D'-1)(D'-2)$ - - - - - -

Substituting these values in (1), (1) becomes a linear D.E. with Constant Coefficients which can be solved as before.

Problems.

(7) Solve

(a)(i) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x$.

Solution: Given D.E. is

$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x \longrightarrow (1)$

(1) is cauchy's homogeneous linear D.E.

Put $x = e^z$. $\Rightarrow z = \log x$.

$\Rightarrow \frac{dz}{dx} = \frac{1}{x}$.

Consider $xD = D'$, $x^2D^2 = D'(D'-1)$

where $D = \frac{d}{dx}$, $D' = \frac{d}{dz}$.

$\therefore (1) \Rightarrow (x^2D^2 - xD + 1)y = \log x$.

$\Rightarrow [D'(D'-1) - D' + 1]y = z$.

$\Rightarrow (D'^2 - 2D' + 1)y = z$.

$\Rightarrow (D'-1)^2 y = z \longrightarrow (2)$

To find C.F.

Consider $(D'-1)^2y = 0$.

A.E. is $(m-1)^2 = 0$ $[D'y = m]$.

$\Rightarrow m = 1, 1$.

$\therefore$ C.F. is $(C_1\cancel{z} + C_2)e^{mz}$

$\Rightarrow$ C.F $= (C_1 z + C_2)e^z \longrightarrow (3)$

To find P.I.

P.I. $= \frac{1}{(D'-1)^2} z$.

$= (1-D')^{-2} z$.

$= (1 + 2D' + 3D'^2 + \cdots) z$.

$= z + 2$.

$\therefore$ P.I. $= z + 2$.

Hence the solution of (1) is

$y =$ C.F. + P.I.

$= (C_1 + C_2 z)e^z + z + 2$

$\Rightarrow \boxed{y = (C_1 + C_2\log x)x + \log x + 2}$

(7)(a)(ii) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 2\log x$

(iii) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x$.

(Ans.) (ii) C.F $m = \pm i$
&(iii)

C.F $= A\cos z + B\sin z$

$= A\cos\log x + B\sin\log x$

P.I. $= \frac{1}{(D'+1)^2} z = \log x$.

(7)(b) $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$.

Solution: Given D.E is

$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x \longrightarrow (1)$

Put $x = e^z \Rightarrow z = \log x \Rightarrow \frac{dz}{dx} = \frac{1}{x}$.

$\therefore (1) \Rightarrow (x^2D^2 + 4xD + 2)y = e^x$ where $D = \frac{d}{dx}$.

Now $xD = D'$,

$x^2D^2 = D'(D'-1)$.

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$\Rightarrow [D'(D'-1)+4D'+2]y = e^{e^z}$.

$\Rightarrow (D'^2+3D'+2)y = e^{e^z} \longrightarrow ①$

C.F. Consider $(D'^2+3D'+2)y=0$.

A.E. is $m^2+3m+2=0$.

$\Rightarrow m = -2, -1$.

$\therefore$ C.F is $c_1 e^{-z} + c_2 e^{-2z}$.

$\Rightarrow$ C.F $= c_1 x^{-1} + c_2 x^{-2} \longrightarrow ③$

$$P.I. = \frac{1}{D'^2+3D'+2} \cdot e^{e^z}$$

$$= \frac{1}{(D'+1)(D'+2)} \cdot e^{e^z}$$

$$= \left[\frac{1}{D'+1} - \frac{1}{D'+2}\right] e^{e^z} \longrightarrow ④$$

Now $\frac{1}{D'+1} e^{e^z} = \frac{1}{D'+1} \cdot e^{-z} \cdot e^{z} \cdot e^{e^z}$.

$$= e^{-z} \cdot \frac{1}{(D'-1)+1} e^{z} \cdot e^{e^z}$$

$$= e^{-z} \cdot \frac{1}{D'} \cdot e^{z} \cdot e^{e^z}$$

$$= e^{-z} \int e^{e^z} d(e^z)$$

$$= x^{-1} \int e^x dx = x^{-1} e^x \longrightarrow ⑤$$

$\frac{1}{D'+2} e^{e^z} = \frac{1}{D'+2} e^{-2z} \cdot e^{2z} \cdot e^{e^z}$.

$$= e^{-2z} \cdot \frac{1}{(D'-2)+2} e^{2z} \cdot e^{e^z}$$

$$= e^{-2z} \frac{1}{D'} e^{2z} \cdot e^{e^z}$$

$$= e^{-2z} \int e^{e^z} \cdot e^z d(e^z)$$

$$= x^{-2} \int e^x \cdot x \, dx \qquad \because z = \log x \ \& \ x = e^z.$$

$$= x^{-2}\left[xe^x - e^x\right]$$

$\therefore P.I. = x^{-1}e^x + x^{-2}(xe^x - e^x)$

$$= \cancel{x^{-1}e^x} + \cancel{x^{-1}e^x} + x^{-2}e^x$$

$$\Rightarrow \boxed{P.I. = x^{-2}e^x}$$

Hence the Complete solution is

$y = c_1 x^{-1} + c_2 x^{-2} + x^{-2} e^x$.

$\Rightarrow y = c_1 x^{-1} + x^{-2}(c_2 + e^x)$

(or) $\boxed{y = \frac{c_1}{x} + \frac{c_2 + e^x}{x^2}}$

---

7 c) $x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^2$.

Solution:

Put $x = e^z \Rightarrow z = \log x$.

$(x^2D^2 - 4xD + 6)y = x^2$.

$[D(D'-1) - 4D'+6]y = e^{2z}$.

$\Rightarrow (D'^2 - 5D' + 6)y = e^{2z} \longrightarrow ①$

C.F. $= c_1 e^{3z} + c_2 e^{2z}$

$= c_1 x^3 + c_2 x^2 \longrightarrow ②$

$$P.I. = \frac{1}{D'^2 - 5D' + 6} \cdot e^{2z}. \qquad \boxed{D = a = 2}$$

$$= \frac{1}{0} e^{2z}$$

$$= \frac{z e^{2z}}{2D-5} \quad \text{since } \boxed{D r = 0}$$

=

$$= \frac{z e^{2z}}{-1}$$

$$\Rightarrow \boxed{P.I. = -z e^{2z}}$$

$$\Rightarrow P.I. = -x^2 \log x \longrightarrow (3)$$

$$y = C.F. + P.I.$$

$$\Rightarrow \boxed{y = c_1 x^3 + c_2 x^2 - x^2 \log x}$$

---

(7) (c) (ii) $x^2 \frac{d^2y}{dx^2} - 7x\frac{dy}{dx} + 12y = x^2$.

Ans: $m = 2, 6$

$C.F. = Ax^2 + Bx^6$.

$P.I. = \frac{-z e^{2z}}{4} = -\frac{x^2}{4}\log x$.

---

7 c) (iii) $(x^2D^2 - xD + 1)y = x^2$

---

7 d) $x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$.

Solution:

$$(x^2D^2 - 2xD - 4)y = x^4 \longrightarrow (1)$$

$$\Rightarrow (D'^2 - 3D - 4)y = e^{4z} \longrightarrow (2)$$

C.F. $m = -1, 4$

$$\therefore C.F. = c_1 e^{-z} + c_2 e^{4z}$$

$$= \frac{c_1}{x} + c_2 x^4 \longrightarrow (3)$$

$$P.I. = \frac{1}{D^2 - 3D - 4} e^{4z} \quad D = a = 4,$$

$$= \frac{z}{2D - 3} e^{4z} \quad \text{since } Dr = 0$$

$$= \frac{z}{5} e^{4z}$$

$$\Rightarrow P.I. = \frac{x^4 \log x}{5} \longrightarrow (4)$$

Solution is $y = C.F. + P.I.$

7e) (i) $x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \sin(\log x)$.

Solution:

$$(x^2D^2 + 4xD + 2)y = \sin(\log x).$$

$$\Rightarrow (D'^2 + 3D' + 2)y = \sin z \longrightarrow (1)$$

$m = -1, -2$

$$\Rightarrow C.F. = Ae^{-z} + Be^{-2z}$$

$$= \frac{A}{x} + \frac{B}{x^2} \longrightarrow (2)$$

$$P.I. = \frac{1}{D'^2 + 3D' + 2} \cdot \sin z. \qquad D'^2 = -1$$

$$= \frac{1}{-1 + 3D + 2} \sin z.$$

$$= \frac{1}{3D + 1} \sin z.$$

$$= \frac{3D - 1}{9D^2 - 1} \sin z. \qquad D'^2 = -1,$$

$$= \frac{3\cos z - \sin z}{-10}$$

$$= \frac{\sin z - 3\cos z}{10}$$

$$\Rightarrow P.I. = \frac{\sin \log x - 3\cos \log x}{10}$$

Solution is $y = C.F. + P.I.$

7e) (ii) $x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 3\sin(2\log x)$.

Ans: $(D'^2 - 5D' + 6)y = 3\sin 2z$.

$m = 3, 2$.

$C.F. = c_1 x^3 + c_2 x^2$.

$$P.I. = \frac{6\sin 2z + 3\cos 2z}{104}$$

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(5)

7f) $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$.

Solution: Given D.E. is

$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$.

$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 12\log x \rightarrow (1)$

$(x^2D^2 + xD)y = 12\log x$.

$\Rightarrow [D'(D'-1) + D']y = 12z$.

$\Rightarrow D'^2 y = 12z \rightarrow (2)$

C.f: $m = 0, 0$.

$\Rightarrow$ C.f $= (Az + B)e^{0z}$

$= A\log x + B \rightarrow (3)$

P.I. $= \frac{1}{D^2} \cdot 12z = 12\frac{1}{D}\int z\,dz$

$= 12 \cdot \frac{1}{D} \cdot \frac{z^2}{2}$

$= 6\int z^2 dz = \frac{6z^3}{3}$.

$= 2z^3$.

$\Rightarrow$ P.I $= 2(\log x)^3$

$\therefore$ Solution is $y =$ C.f. + P.I.

7g). $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = (x+1)^2$ (or) $(1+x)^2$.

Solution:

$[x^2D^2 - 3xD + 4]y = x^2 + 2x + 1. \rightarrow (1)$

$[D'(D'-1) - 3D' + 4]y = e^{2z} + 2e^z + 1$.

$\Rightarrow (D'^2 - 4D' + 4)y = e^{2z} + 2e^z + 1$.

$\Rightarrow (D'-2)^2 y = e^{2z} + 2e^z + 1 \rightarrow (2)$

C.f: $m = 2, 2$

$\Rightarrow$ C.f $= (c_1 z + c_2)e^{2z}$.

$= (c_1\log x + c_2)x^2 \rightarrow (3)$

P.I. $= \frac{1}{(D-2)^2}(e^{2z} + 2e^z + 1)$.

$= \frac{1}{(D-2)^2}e^{2z} + 2\frac{1}{(D-2)^2}e^z + \frac{1}{(D-2)^2}e^{0z}$.

$\boxed{D=2}$ $\boxed{D=1}$ $\boxed{D=0}$

$= \frac{z}{2(D-2)}e^{2z} + 2\cdot\frac{1}{(-1)^2}e^z + \frac{1}{(-2)^2}e^{0z}$.

(since Dr = 0)

$= \frac{z^2}{2}e^{2z} + 2e^z + e^z$

(since Dr = 0)

$\therefore \boxed{\text{P.I.} = \frac{x^2(\log x)^2}{2} + 2x + \frac{1}{4}}$

$\therefore$ Solution is $y =$ C.f. + P.I.

7) h) $(D^2 - xD - 3)y = x^2\log x$.

Solution: $D = \frac{d}{dx}$, $\theta = \frac{d}{dz}$.

$[\theta(\theta-1) - \theta - 3]y = e^{2z} \cdot z$

$\Rightarrow (\theta^2 - 2\theta - 3)y = e^{2z} \cdot z \rightarrow (1)$

C.f: $m = -1, 3$

$= Ae^{-z} + Be^{3z} = \frac{A}{x} + Bx^3 \rightarrow (2)$

P.I. $= \frac{1}{\theta^2 - 2\theta - 3} ze^{2z}$

$= e^{2z}\frac{1}{(\theta+2)^2 - 2(\theta+2) - 3}z$ $\quad$ $D+a = D+2$

$= e^{2z}\frac{1}{\theta^2 + 2\theta - 3}z$.

$= e^{2z}\frac{1}{-3\left[1 - \left(\frac{\theta^2 + 2\theta}{3}\right)\right]}z$

$= \frac{-e^{2z}}{3}\left[1 + \theta^2/3 + \frac{2\theta}{3}\right](z)$

$= \frac{e^{2z}}{3}(z + 0 + 2/3) = -x^2/3(\log x + 2/3)$

$y =$ C.f + P.I $\rightarrow$ solution

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7 (i) $(x^2D^2 - 2xD + 4)y = x^2 + 2\log x$.

Soln: (i) $(\theta^2 - 3\theta + 4)y = e^{2z} + 2z \rightarrow (1)$

C.F. $m = \frac{3 \pm i\sqrt{7}}{2}$

$= e^{\frac{3}{2}z}\left(c_1 \cos\frac{\sqrt{7}}{2}z + c_2 \sin\frac{\sqrt{7}}{2}z\right)$

$= x^{3/2}\left(c_1 \cos\frac{\sqrt{7}}{2}\log x + c_2 \sin\frac{\sqrt{7}}{2}\log x\right)$

$P.I._1 = \frac{1}{\theta^2 - 3\theta + 4} e^{2z} = \frac{e^{2z}}{2}$

$= x^2/2 \rightarrow (2)$

$P.I._2 = \frac{1}{\theta^2 - 3\theta + 4} \cdot 2z$

$= 2\frac{1}{4}\left[1 - \left(\frac{\theta^2 - 3\theta}{4}\right)\right]^{-1}(z)$

$= \frac{1}{2}(z + 3/4)$

$= \frac{1}{2}(\log x + 3/4) \rightarrow (3)$

$P.I. = P.I._1 + P.I._2$

$= \frac{x^2}{2} + \frac{1}{2}\left(\log x + \frac{3}{4}\right)$

$y = C.F + P.I$ is the solution.

7 (j) x^2 (ii) $(x^2D^2 - 2xD - 4)y = x^2 + 2\log x$

(ii)

7 (j) $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = \cos(3\log x)$

Ans. $(D^2 - 3D - 4)y = \cos 3z \rightarrow (1)$

$CF = \frac{c_1}{x} + c_2 x^4 \rightarrow (2)$

$P.I = -\frac{(9\sin 3z + 13\cos 3z)}{250}$

7 (k) $(x^2D^2 - 4xD + 6)y = \frac{42}{x^4}$.

Soln: $(\theta^2 - 5\theta + 6)y = 42e^{-4z} \rightarrow (1)$

$CF = Ax^2 + Bx^3 \rightarrow (2)$

$P.I. = \frac{1}{\theta^2 - 5\theta + 6} \cdot 42e^{-4z}$

$= e^{-4z} = x^{-4} \rightarrow (3)$

$\therefore$ Solution is

$y = C.F + P.I$

7 (l) $(x^2D^2 + 4xD + 2)y = x\log x$.

Soln: $(\theta^2 + 3\theta + 2)y = e^z \cdot z$

$CF = \frac{c_1}{x} + \frac{c_2}{x^2}$.

$P.I. = \frac{e^z}{6}(z - 5/6)$

$= \frac{x}{6}(\log x - 5/6)$

7 (m) $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} = x + 1$.

Soln: $(\theta^2 - 4\theta)y = e^z + 1 \rightarrow (1)$

$CF = A + Bx^4 \rightarrow (2)$

$P.I. = \frac{1}{\theta^2 - 4\theta} e^z + 1$.

$= \frac{1}{\theta^2 - 4\theta} \cdot e^z + \frac{1}{\theta(\theta - 4)} e^{0z}$

$= -\frac{e^z}{3} - \frac{1}{4}z$

$P.I = -\frac{x}{3} - \frac{\log x}{4}$

(7)(n) $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 5y = x^2\sin(\log x)$

soln: $(D^2 - 4D + 5)y = e^{2z}\sin z \rightarrow (2)$

C.F. $m = 2 \pm i$

$= e^{2z}(c_1\cos z + c_2\sin z)$

$= x^2(c_1\cos\log x + c_2\sin\log x) \rightarrow (3)$

$P.I. = \frac{1}{D^2-4D+5} e^{2z}\sin z$

$= e^{2z}\frac{1}{D^2+1}\sin z$

e^{iz}

$= I.P. \text{ of } e^{2z}\frac{1}{D^2+1}e^{iz}$

$= e^{2z} I.P. \text{ of } \frac{1}{2i}\frac{1}{D-i}e^{iz}$

since $\frac{1}{D^2+1} = \frac{1}{(D+i)(D-i)}$

$= \cancel{e^{2z} IP\left[-\frac{i}{2}z(\cos\right.}$

$= e^{2z} I.P. \text{ of } \frac{1}{2i} z e^{iz}$

$= e^{2z} IP\left[-\frac{i}{2}z(\cos z + i\sin z)\right]$

$= -\frac{e^{2z} z\cos z}{2}$

$= -\frac{x^2}{2}\log x\cos(\log x)$.

(7)(o) The radial displacement u in a rotating disc at a distance r from the axis is given by

$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} + kr = 0$ (1)

find the displacement

soln: $r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u = -kr^3 \rightarrow (2)$.

$(D^2-1)u = -ke^{3z}$.

CF $c_1 r + \frac{c_2}{r}$

$P.I. = \frac{1}{D^2-1}(-ke^{3z})$

$= -\frac{k}{8}e^{3z} = -\frac{k}{8}r^3$

$y = Cf + P.I.$

(7)(p), $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^2\cos\log x$.

soln: $(D^2-5D+6)y = e^{2z}\cos z$

$CF = c_1x^2 + c_2x^3$

$P.I. = \frac{1}{D^2-5D+6}e^{2z}\cos z$

$= e^{2z}\frac{1}{-D}\cos z$

$= -\frac{e^{2z}}{2}(\sin z + \cos z)$

$= -\frac{x^2}{2}[\sin\log x + \cos\log x]$

⑧

⑦ ⑨ $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x\sin(2\log x)$

Soln: $(\theta^2-3\theta-4)y = e^z\sin 2z$

$CF = c_1x^{-1}+c_2x^4$

$P.I. = \frac{1}{\theta^2-3\theta-4}e^z\sin 2z$

$= e^z\cdot\frac{1}{-D-10}\sin 2z$

$= \frac{e^z}{104}(\cos 2z - 10\sin 2z)$

where $z = \log x$

$= \frac{e^z}{-108}(36z^2+42z+23)$

$= \frac{-x}{108}(36(\log x)^2+42\log x+23)$

⑦ ⑧ $x^2\frac{d^2y}{dx^2} - 6x\frac{dy}{dx} + 6y = \frac{1}{x}[2(\log x)^2 - 3\log x + 4]$.

Soln: $(D^2-5D+6)y = e^{-z}(2z^2+3z+4)$

$CF = c_1x^2 + c_2x^3$.

$P.I. = \frac{1}{D^2-5D+6}e^{-z}(2z^2+3z+4)$

$= e^{-z}\cdot\frac{1}{D^2-7D+12}(2z^2+3z+4)$

$= \frac{e^{-z}}{12}\left(2z^2-\frac{2z}{3}+\frac{17}{4}\right)$.

$= \frac{1}{144x}\left[24(\log x)^2 - 8\log x + 51\right]$

⑦ ⑧ $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x[2(\log x)^2+3\log x+1]$

Soln: $(\theta^2-3\theta-4)y = e^z(2z^2+3z+1)$

$CF = c_1x^{-1}+c_2x^4$.

$P.I. = \frac{1}{\theta^2-3\theta-4}e^z(2z^2+3z+1)$.

$= e^z\cdot\frac{1}{\theta^2-\theta-6}(2z^2+3z+1)$

$= \frac{e^z}{-6}\left[1+\frac{D^2}{6}-\frac{D}{6}+\frac{D^2}{36}\right](\ )$

$= \frac{e^z}{-216}[7D^2-6D+36](\ )$

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12/09/2020

II B.Sc Mathematics

DIFFERENTIAL EQUATIONS

Legendre's Linear Equation.

Consider an equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + k_1 (ax+b)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \dots + k_n y = X \rightarrow (1)$$

where $k_1, k_2, \dots, k_n$ are constants and X is a function of x, is called Legendre's linear equation.

(1) is a D.E. with variable coefficients.

Such equations can be reduced to linear equations with constant coefficients by suitable substitutions.

Put $ax+b = e^z$

$\Rightarrow z = \log(ax+b)$. $D' = \frac{d}{dz}$

Then if $D = \frac{d}{dx}$,

$z = \log(ax+b)$

$\Rightarrow \frac{dz}{dx} = \frac{a}{ax+b}$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{dy}{dz} \cdot \frac{a}{ax+b}$$

$$= \frac{a}{ax+b} \frac{dy}{dz}$$

$$\Rightarrow (ax+b)\frac{dy}{dx} = a\frac{dy}{dz}.$$

$$\Rightarrow [(ax+b)D]y = [aD']y$$

$$\Rightarrow \boxed{(ax+b)D = aD'}$$

Also $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$$= \frac{d}{dx}\left[\frac{a}{ax+b}\frac{dy}{dz}\right]$$

$$= -\frac{a^2}{(ax+b)^2}\frac{dy}{dz} + \frac{a}{ax+b}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= -\frac{a^2}{(ax+b)^2}\frac{dy}{dz} + \frac{a}{ax+b}\frac{d^2y}{dz^2}\cdot\left(\frac{a}{ax+b}\right).$$

$$= \frac{a^2}{(ax+b)^2}\left[\frac{d^2y}{dz^2} - \frac{dy}{dz}\right].$$

$$\Rightarrow (ax+b)^2\frac{d^2y}{dx^2} = a^2\left[D'^2 - D'\right]y$$

$$\Rightarrow \left[(ax+b)^2 D^2\right]y = a^2\left[D'^2 - D'\right]y.$$

$$\Rightarrow \boxed{(ax+b)^2 D^2 = a^2 D'(D'-1)}$$

|||ly $(ax+b)^3 D^3 = a^3 D'(D'-1)(D'-2)$ ......

Hence the substitutions are

$$\boxed{\begin{array}{l} D = d/dx, \quad D' = d/dz, \\ (ax+b) = e^z, \quad z = \log(ax+b) \\ (ax+b)D = aD', \quad (ax+b)^2 D^2 = a^2 D'(D'-1). \end{array}}$$

Substituting these values in (1), (1) becomes a linear D.E. with constant coefficients which can be solved easily.

If $n=2$, we get (1) as

$$(ax+b)^2\frac{d^2y}{dx^2} + (ax+b)p_1\frac{dy}{dx} + p_2 y = X$$

where p_1, p_2 are constants and X is any function of x.

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Problems-

(8) Solve

(a) $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin[\log(1+x)]$.

Solution: Given D.E is

$$(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin[\log(1+x)] \rightarrow (1)$$

(1) is a Legendre's linear equation with variable coefficients.

$\therefore$ Put $(1+x) = e^z \Rightarrow z = \log(1+x)$

$(ax+b) = e^z$
Here $a=1$, $b=1$.

$\Rightarrow \frac{dz}{dx} = \frac{1}{1+x}$.

$D = \frac{d}{dx}$, $D' = \frac{d}{dz}$.

Also

$(1+x)\frac{dy}{dx} = D'y$.

& $(1+x)^2\frac{d^2y}{dx^2} = D'(D'-1)y$.

$\therefore (1) \Rightarrow D'(D'-1)y + D'y + y = 2\sin z$

$\Rightarrow [(D')^2 - D' + D' + 1]y = 2\sin z$

$\Rightarrow (D'^2+1)y = 2\sin z \rightarrow (2)$

(2) is a linear equation with constant coefficients.

To find C.F. Consider

$(D'^2+1)y = 0 \rightarrow (3)$

A.E is $m^2+1 = 0 \Rightarrow m^2 = \pm i$
$= \alpha \pm i\beta$

$\alpha = 0$, $\beta = 1$.

$\therefore$ C.F. $= e^{\alpha z}[A\cos\beta z + B\sin\beta z]$.

$\Rightarrow$ C.F $= c_1\cos z + c_2\sin z$.
$= c_1\cos\log(1+x) + c_2\sin\log(1+x) \rightarrow (4)$

To find P.I.

$$P.I. = \frac{1}{D'^2+1}\cdot 2\sin z. \qquad D'^2 = a^2 = -1.$$

$$= 2\cdot\frac{1}{(Dr=0)}\cdot\sin z$$

$$= 2\frac{z}{2D'}\sin z.$$

$$= z\int\sin z\,dz$$

$$= -z\cos z. = -z\cos\log(1+x) \rightarrow (5)$$

Now the solution is

$y = C.F + P.I.$

By (4) & (5),

$$\boxed{y = c_1\cos[\log(1+x)] + c_2\sin[\log(1+x)] - \log(1+x)\cos\log(1+x)}.$$

(b) $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = \sin[2\log(1+x)]$.

Hint: R.H.S. $= \sin 2z$.

Proceed as (8) (a)

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8) (c) $(5+2x)^2\frac{d^2y}{dx^2} - 6(5+2x)\frac{dy}{dx} + 8y = 6x.$

Solution:

Put $5+2x = e^z \Rightarrow z = \log(5+2x)$

$\Rightarrow \frac{dz}{dx} = \frac{2}{5+2x}$.

$D = \frac{d}{dx}, \quad \theta = \frac{d}{dz}$

$5+2x = e^z$
$2x = e^z - 5$
$x = \frac{e^z-5}{2}$

$(5+2x)D = 2D'$

$(5+2x)^2 D^2 = 4D'(D'-1)$

∴ The given equation becomes

$[4D'(D'-1) - 12D' + 8]y = 6\left(\frac{e^z-5}{2}\right)$.

$\Rightarrow (4D'^2 - 16D' + 8)y = \left(\frac{e^z-5}{2}\right)6$. $\rightarrow$ (1)

To find C.F.

A.E. is $4m^2 - 16m + 8 = 0$.

$\Rightarrow 2m^2 - 8m + 4 = 0$

$\Rightarrow m^2 - 4m + 2 = 0$

$m = \frac{4 \pm \sqrt{16-8}}{2}$

$= \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$

~~$C.F. = e^{2z}[C_1 \cos\sqrt{2}z + C_2 \sin\sqrt{2}z]$~~ $\rightarrow$ (2)

$C.F. = Ae^{(2+\sqrt{2})z} + Be^{(2-\sqrt{2})z}$.

$= e^{2z}\left(Ae^{\sqrt{2}z} + Be^{-\sqrt{2}z}\right)$

$= (5+2x)^2\left[A(5+2x)^{\sqrt{2}} + B(5+2x)^{-\sqrt{2}}\right]$

$\rightarrow$ (2).

$P.I. = \frac{1}{4D'^2 - 16D' + 8}\, 6\left[\frac{e^z}{2} - \frac{5}{2}e^{0z}\right]$.

$= 6\left[\frac{1}{4-16+8}\frac{e^z}{2} - \frac{5}{2}\frac{1}{0-0+8}e^{0z}\right]$

$(D'=1) \qquad (D'=0)$

$= 6\left[-\frac{e^z}{8} - \frac{5}{16}\right]$.

$= 6\left[-\frac{(5+2x)}{8} - \frac{5}{16}\right] = 6\left[-\frac{5}{8} - \frac{2x}{8} + \frac{5}{16}\right]$

$= 6\left[-\frac{x}{4} - \frac{10+5}{16}\right] = \left[-\frac{x}{4} - \frac{15}{16}\right]6$

$= -\frac{3}{2}x - \frac{45}{8} \rightarrow$ (3)

Solution is $y = C.F. + P.I.$
using (2) & (3).

8) (d) $(2x+3)^2\frac{d^2y}{dx^2} - (2x+3)\frac{dy}{dx} - 12y = 6x.$

8) (e) $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\log(1+x).$

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